Computational Issues in Hemodynamics

http://www.ann.jussieu.fr/pironneau

Olivier Pironneau[1]

[1]with input from **Frédéric Hecht**
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CASTS-LJLL Taipei, May 2014

Blood Flow: Motivations and Modeling

Aortic Flow

Goals

- Understand blood flow, red cells, cholesterol, the heart...
- Understand aneurisms, the effects of stents, heart valves...
- Improve stents

Modeling

- Fluid-structure interactions with large displacements and contacts in complex medium!
- Ignore the medium outside the heart, arteries, blood vessels...
- Assume perfect Newtonian fluid for the blood.
- Assume small displacement visco-elastic model for the blood vessels.

[1]F. Usabiaga, J. Bell, R. Buscalioni, A. Donev, T. Fai, B. Griffith, and C. Peskin. Staggered schemes for fluctuating hydrodynamics. Multiscale Model Sim. 10:1369-1408, 2012.

Linear Models for the Wall of the Arteries

A hierarchy of approximations for the displacement $\vec{d}$ of the structure:

- Koiter's linear elasticity shell model: $h << 1$ with small displacements,
- $+$ pre-stress and visco-elastic terms T, a, b: empty arteries are flat
- Assume normal displacement η only
- Add damping terms C to account for loss of energy

Then on the mean position Σ the model reduces to [1] $\vec{d} = \eta\vec{n}$:

$$\rho_s h\partial_{tt}\eta - \nabla\cdot(\mathbf{C}\nabla\partial_t\eta) - \nabla\cdot(\mathbf{T}\nabla\eta) + a\partial_t\eta + b\eta = f^s, \;\; \eta, \partial_t\eta \text{ given}|_{t=0}$$

f^s is the external normal force, i.e. $-\sigma^{\mathbf{s}}{}_{nn}$.

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[1]F. Nobile and C. Vergana, an effective fluid-structure interaction formulation for vascular dynamics by generalized robin conditions. SIAM J. Sci. Comp. Vol. 30, No. 2, pp. 731-763 (2008)

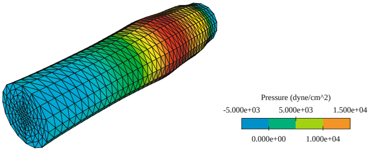
Surface Pressure Model for the Arteries

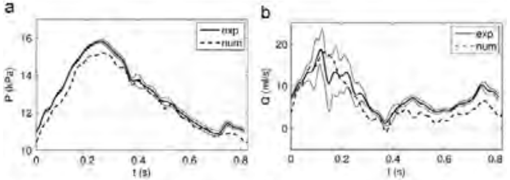
- Assuming $[h, T, C, a] << b$ leads to the surface pressure model

$$\eta_{|0}, \partial_t \eta_{|0} \text{ given, } \rho_s h \partial_{tt}\eta - \nabla\cdot(T\nabla\eta + C\partial_{nt}\eta) + a\partial_t\eta + b\eta = f^s$$

$$\boxed{-\sigma^s{}_{nn} = b\eta, \quad \text{with} \quad b = \frac{Eh\pi}{A(1-\xi^2)}}$$

A: vessel's cross section, E: Young modulus, ξ: Poisson coeff. e.g. (MKS)
$E = 3MPa, \quad \xi = 0.3, h = 0.001, \rho^f = 9.81\ 10^6 \Rightarrow b = 3.3 10^7 ms^{-2} \Rightarrow \frac{b}{\rho^f} = 3.37$





Gives displacements $\approx 0.1\ 10^{-3}$m and flow rates $\approx 2\ 10^{-5}$m^{3}s^{-1}

[2]L. Formaggia, A. Quarteroni, A. Veneziani (eds), Cardio-Vascular Mathematics Springer MS&A (2009)

Fluid Equations

Navier-Stokes equations in a moving domain $\Omega(t)$: for all v, q

$$\rho^f(\partial_t \vec{u} + u \cdot \nabla u) + \nabla p - \mu \Delta \vec{u} = 0, \quad \nabla \cdot \vec{u} = 0$$

Continuity on Σ of velocities : $\vec{u} = \vec{n}\partial_t \eta$,
Continuity of normal stress : $\vec{n} \cdot (\mu(\nabla u + \nabla u^T) - p)\vec{n} = -\sigma^s_{nn} := b\eta$
And tangential stress ?? $\sigma^s_{n\tau} = \sigma^f_{n\tau}$

ALE Navier-Stokes [1][2]: Assume $\Omega_t = \mathcal{A}_t(\Omega_0)$ with $\mathcal{A}_t : x_0 \to x_t := \mathcal{A}_t(x_0)$. Let $u_\tau(x,t) = u(\mathcal{A}_t(\mathcal{A}_\tau^{-1}(x)), t), \ \forall x \in \Omega_\tau$ Then

$$\partial_t \vec{u}_\tau + (\vec{u}_\tau - \vec{c}_\tau) \cdot \nabla \vec{u}_\tau + \nabla p - \nu \Delta \vec{u}_\tau = 0, \quad \nu = \frac{\mu}{\rho_f}, \ \frac{p}{\rho^f} \to p$$

$$\nabla \cdot \vec{u}_\tau = 0, \quad + \text{B.C. with} \quad c_\tau(x) = -\frac{\partial}{\partial t}\left[\mathcal{A}_t(\mathcal{A}_\tau^{-1}(x))\right]|_{t=\tau}$$

[1] A. Decoene & B. Maury Moving Meshes with freefem++. J. Numer Math (20)3-4, p195-214(2013).

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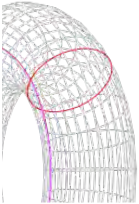
Transpiration Conditions for the Fluid

Σ_t the moving boundary, Σ a reference bdy: $\Sigma_t = \{x + \eta\vec{n} : x \in \Sigma\}$

$u(x+\eta\vec{n}) = \vec{n}\partial_t\eta(x),\ x \in \Sigma \ \Rightarrow\ u + \eta\dfrac{\partial u}{\partial n} = \vec{n}\partial_t\eta + o(\eta)$ on Σ

On a torus (r,R), $u \times n = 0,\ \nabla\cdot u = 0 \Rightarrow n\cdot\dfrac{\partial u}{\partial n} = (1+\dfrac{r}{R}\cos^2\theta)\dfrac{u\cdot n}{r} \Rightarrow$

$$u(x+\eta\vec{n})\cdot n = o(\eta) = u\cdot n\left(1+\frac{\eta}{r}(1+\frac{r}{R}\cos^2\theta)\right) = \partial_t\eta$$



Hence now the domain is fixed and with precision $O(\frac{\eta}{r})$

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Remark On a torus (r,R), $\sigma^f_{nn} = p + 2(1+\frac{r}{R}\cos^2\theta)\frac{\mu}{r}u\cdot n$. Hence

$$u\cdot n = \partial_t\eta/(1+\frac{\eta}{r}(1+\frac{r}{R}\cos^2\theta)),\quad p \approx b\eta + \frac{2\mu\partial_t\eta}{r}(1+\frac{r}{R}\cos^2\theta - \frac{\eta}{r})$$

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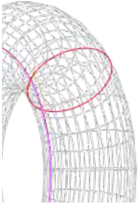

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Energy Balance is Disturbed

Variational Form & Conservation of Energy in a moving domain

$$\int_{\Omega(t)} [\hat{u}(\partial_t u + u \cdot \nabla u - f^s) - p\nabla \cdot \hat{u} - \hat{p}\nabla \cdot u + \frac{\nu}{2}(\nabla u + \nabla u^T) : (\nabla \hat{u} + \nabla \hat{u}^T)] = 0$$

An energy conservation is obtained by taking $\hat{u} = u$ and $\hat{p} = -p$,

$$\partial_t \int_{\Omega(t)} \frac{u^2}{2} + \frac{\nu}{2} \int_\Omega |\nabla u + \nabla u^T|^2 = \int_\Omega f^s \cdot u$$

as $\partial_t \int_{\Omega(t)} u \cdot w = \int_{\Omega(t)} \partial_t (u \cdot w) + \int_{\partial\Omega} v \; u \cdot w$ and $\int_\Omega (u\nabla u) \cdot u = \int_{\partial\Omega} u \cdot n \frac{u^2}{2}$. If w is the normal velocity of $\partial\Omega$ the variational formulation should be:

$$\int_\Omega [\hat{u} \cdot (\partial_t u + u\nabla u) - p\nabla \cdot \hat{u} - \hat{p}\nabla \cdot u \quad + \frac{\nu}{2}(\nabla u + \nabla u^T) : (\nabla \hat{u} + \nabla \hat{u}^T)]$$

$$- \int_{\partial\Omega} \frac{w}{2} u \cdot \hat{u} = \int_\Omega f^s \cdot \hat{u}$$

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$$- \int_{\partial\Omega}\frac{w}{2}u\cdot\hat{u} = \int_{\Omega} f^s\cdot\hat{u}$$

Variational Formulation with Pressure Conditions

In 1986 it was shown in [1][2] showed that

$$\alpha(u,\hat u)+\nu(\nabla\times u,\nabla\times\hat u)=(f,\hat u)+\int_{\partial\Omega}p_\Gamma\hat u_n$$

is well posed in $J_0(\Omega)=\{v\in H^1(\Omega)\ :\ \nabla\cdot u=0,\ v\times n|_{\partial\Omega}=0\}$

$$\Leftrightarrow\quad \alpha u-\nu\Delta u+\nabla p=f,\ \nabla\cdot u=0,\ u\times n|_{\partial\Omega}=0,\ p|_{\partial\Omega}=p_\Gamma,$$

Consequently find $[\vec u,p,\eta]$ with $u\times n=0$ and $\forall\hat u,\hat p,\hat\eta$ with $\hat u\times n=0$,

$$\int_\Omega[\hat u\cdot(\partial_t u-u\times\nabla\times u)-p\nabla\cdot\hat u-\hat p\nabla\cdot u+\nu\nabla\times u\cdot\nabla\times\hat u]$$
$$+\int_{\partial\Omega\backslash\Gamma}b[\eta\hat u\cdot n+\hat\eta(u\cdot n-\partial_t\eta)]=\int_\Gamma p_\Gamma\hat u\cdot n\ \Rightarrow$$
$$\partial_t u-u\times\nabla\times u+\nabla p-\nu\Delta u=0,\ p=b\eta,\ u\cdot n=\partial_t\eta \text{ or } p_{|\Gamma}=p_\Gamma$$

[1] O. Pironneau: Conditions aux limites sur la pression pour les eq. Navier-Stokes. CRAS 303,i, p403-406. 1986.
[2] O. Pironneau: Finite Elements for Fluids. J. Wiley 1989.

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$$+\int_{\partial\Omega\setminus\Gamma} b[\eta\hat{u}\cdot n + \hat{\eta}(u\cdot n - \partial_t\eta)] = \int_\Gamma p_\Gamma \hat{u}\cdot n \ \Rightarrow$$
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Time discretized Variational Formulation

Then $p = b\eta,\ u = \vec{n}\partial_t\eta,\ \Rightarrow\ bu\cdot n = \partial_t p \Rightarrow p^{m+1} - p^m = \delta t b u^{m+1}\cdot n$

$$\int_\Omega \Big(\partial_t u - u\times\nabla\times u + \nabla p)\cdot\hat{u} + \nu\nabla\times u\cdot\nabla\times\hat{u}\Big) = 0$$

Variational formulation: $\forall\hat{u}\in H^1(\Omega)^3,\ \hat{p}\in L^2(\Omega)$,

$$\boxed{\begin{aligned}&\int_\Omega\Big[\hat{u}\cdot(\frac{u^{m+1}-u^m}{\delta t} - u^{m+\frac12}\times\nabla\times u^m) - p^{m+1}\nabla\cdot\hat{u} - \hat{p}\nabla\cdot u^{m+\frac12}\\ &+\nu\nabla\times u^{m+\frac12}\cdot\nabla\times\hat{u}\Big] + \int_{\partial\Omega}[\frac{1}{\epsilon}u^{m+\frac12}\times n\cdot\hat{u}\times n + \hat{u}\cdot(u^{m+\frac12}b\delta t + p^m\vec{n})]\\ &+\int_{\partial\Omega}[h\partial_{tt}u\cdot\hat{u} + \nabla\hat{u}^T\mathbf{C}\nabla\partial_t u + u^{m+\frac12}\mathbf{T}\nabla\hat{u} + \big(a{-}2(1+\frac{r}{R}\cos^2\theta)\frac{\nu}{r}\big)\partial_t u\cdot\hat{u}] = 0\end{aligned}}$$

$$\hat{u} = u^{m+\frac12}, \hat{p} = -p^{m+1} \Rightarrow \|u^{m+1}\|_0^2 + \nu\delta t\sum_{k\le m}\Big(\|\nabla u^{k+\frac12}\|_0^2 + b\delta t^2\int_{\partial\Omega}|u^{k+\frac12}|^2\Big)$$

$$+\frac{1}{2b}\int_{\partial\Omega}[\sum_{k\le m}(p^{k+1}-p^k)^2 - {p^{m+1}}^2 + {p^0}^2] = \|u^0\|_0^2$$

RUN

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Potential Blood Flow

By putting $\hat{u} = \nabla q,\ \hat{p} = 0,\ \hat{\eta} = 0$ in the variational formulation,

$$-\Delta p = 0 \text{ in } \Omega,\ \frac{\partial p}{\partial n}|_\Sigma = -\partial_t u \cdot n,\quad p|_\Gamma = p_\Gamma \ \Rightarrow$$
$$-\Delta p = 0 \text{ in } \Omega,\ \partial_{tt} p + b\frac{\partial p}{\partial n} = 0 \text{ on } \Sigma,\quad \Rightarrow\ \partial_{tt} p_\Sigma - b\Delta_\Sigma p_\Sigma = 0$$

where $-\Delta_\Sigma$ is the Steklov-Poincaré op. Resonance at $\sqrt{b\lambda_{(-\Delta_\Sigma)}}$.

Example Blood wave speed $v \approx 5m/s$. A pressure drop in $\cos^2(k\pi t)$ gives resonance in $\Omega = (0,L)\times(0,R)$ only if $L = k\pi v$. Then $\lambda \approx \frac{\sqrt{bR}}{v}$ so first eigenvalue $\lambda_1 = 36cm$ when R=1.3cm and h=0.1cm, leading to b=3.37.

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Implementation with freefem++

$$p(t) = p(0) + bU(t) \text{ with } U(t) = \int_0^t u \cdot n(s) ds \ \Rightarrow U^{m+1} = U^m + u^{m+1} \mathrm{d}t$$

$$\int_\Omega \left[\hat{u} \cdot \left(\frac{u^{m+1} - u^m}{\delta t} - u^{m+1} \times \nabla \times u^m\right) - p^{m+1} \nabla \cdot \hat{u} - \hat{p} \nabla \cdot u^{m+\frac{1}{2}}\right.$$

$$\left. + \nu \nabla \times u^{m+1} \cdot \nabla \times \hat{u}\right] + \int_{\partial\Omega} [\frac{1}{\epsilon} u^{m+1} \times n \cdot \hat{u} \times n + b\hat{u} \cdot \vec{n}(u^{m+1} \delta t + U^m) \cdot \vec{n}] = 0$$

```
problem bb([u,v,w,p],[uh,vh,wh,ph], solver=UMFPACK)
        = int3d(th)((u*uh+v*vh+w*wh)/dt2
                  + nu*(dx(u)*dx(uh)+dy(u)*dy(uh)+dz(u)*dz(uh)        // rot u.rot uh
                      + dx(v)*dx(vh)+dy(v)*dy(vh) +dz(v)*dz(vh)
                      + dx(w)*dx(wh)+dy(w)*dy(wh) +dz(w)*dz(wh)
                      -(dx(u)+dy(v)+dz(w))*(dx(uh)+dy(vh)+dz(wh))
                   ) - (dx(uh)+dy(vh)+dz(wh))*p - (dx(u)+dy(v)+dz(w))*ph //-ph div u - p div uh
- v*(dx(vold)-dy(uold))*uh -w*(dy(wold)-dz(vold))*vh -u*(dz(uold)-dx(wold))*wh // u x rot u
+ w*(dz(uold)-dx(wold))*uh +u*(dx(vold)-dy(uold))*vh + v*(dy(wold)-dx(vold))*wh
                   )
        - int3d(th)( (uold*uh + vold*vh + wold*wh)/dt2 )              // u^m/dt
        + int2d(th,1)((     (u*N.y-v*N.x)*(uh*N.y-vh*N.x)             // uxn.uhxn/eps
                          + (v*N.z-w*N.y)*(vh*N.z-wh*N.y)
                          + (w*N.x-u*N.z)*(wh*N.x-uh*N.z)    )/eps)
        + int2d(th,1)(b*dt2*(u*N.x+v*N.y+w*N.z)*(uh*N.x+vh*N.y+wh*N.z)) // b dt u.n uh.n
        + int2d(th,1)(b*(Uold*N.x+Vold*N.y+Wold*N.z)*(uh*N.x+vh*N.y+wh*N.z)) // b n. int_0^t u
        - int2d(th,2)(p0*wh ) - int2d(th,3)(p1*vh )
        + on(2,3,u=0) + on(2,v=0) +on(3,w=0);
```

Convergence (paper with T. Chacon et al.)

Lemma If Ω is $\mathcal{C}^{1,1}$ or polyhedric and $u_0 \in L^2(\Omega)^3$, $p_0 \in H^{1/2}(\Sigma)$, then the weak solution of the continuous problem verifies $u \in L^2(\mathbf{H}^2)$, $\partial_t u \in L^2(\mathbf{L}^2)$, $p \in L^2(H^1)$, and $u \times n = 0$ in $L^2(L^4(\Sigma))$, $\partial_t p = bu \cdot n$ in $L^2(H^{1/2}(\Sigma))$, $p(0) = p_0$

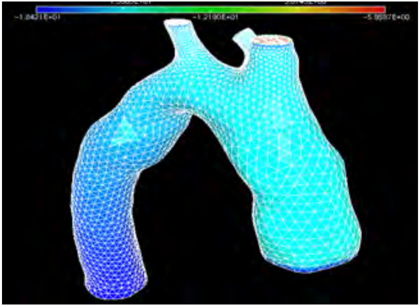
Theorem The solution of the time discretized variational problem satisfies

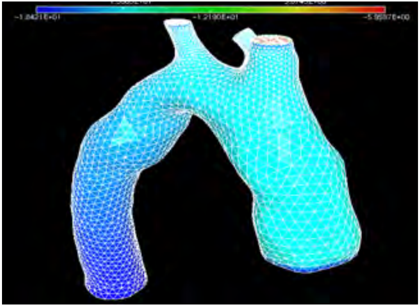
$$\|u_\delta\|_{L^\infty(\mathbf{L}^2)} + \sqrt{\nu}\,\|u_\delta\|_{L^2(\mathbf{H}^1)} + \quad b\,\|\delta t \sum_{k=1}^{n+1} u^k \cdot n\|_{L^\infty(\mathbf{L}^2(\Sigma))}$$
$$\leq C\left(\|u_0\|_{0,2,\Omega} + \frac{1}{\sqrt{\nu}}\|p_0\|_{L^2(\Sigma)}\right)$$

Theorem If Ω is simply connected, $\exists$ subsequence $(u_{\delta'}, p_{\delta'})$ which converges to the continuous problem in $L^2(\mathbf{W}) \times H^{-1}(L^2)$ where

$$\mathbf{W} \quad = \quad \{w \in L^2(\Omega)\,|\,\nabla \times w \in L^2(\Omega),\ \nabla \cdot w \in L^2(\Omega),\ n \times \mathbf{w}_{|_\Sigma} = \mathbf{0}\}.$$

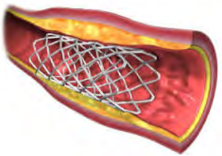
Simulation of an aortic bend

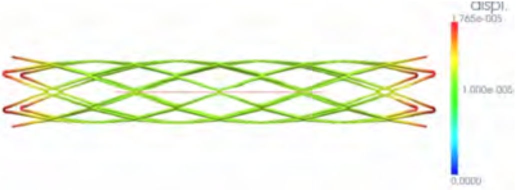




Optimal Stents in the Context of Surface Pressure Models

A stent tuned to the patient? e.g. $\min_{b(x)} J = \int_{\Sigma\times(0,T)} F(p)\mathrm{d}x\mathrm{d}t$:





S. Canic[1] (with freefem++)

Subject to

$$\int_\Omega [\hat u\cdot(\frac{u^{m+1}-u^m}{\delta t} - u^{m+1}\times\nabla\times u^m) - p^{m+1}\nabla\cdot\hat u - \hat p\nabla\cdot u^{m+1}]$$
$$+\int_\Omega \nu\nabla\times u^{m+1}\cdot\nabla\times\hat u + \int_\Sigma (u^{m+1}b\delta t + p^m n)\cdot\hat u = -\int_\Gamma p_\Gamma \hat u_n$$
$$\forall \hat u\in V_h, \hat p\in Q_h \text{ with } \hat u\times n|_\Gamma = 0$$

For instance $F = |p|^4$ will minimize the pressure peak on the surface.

[1] J. Tambaca, S. Canic, M. Kosor, R.D. Fish, D. Paniagua. Mechanical Behavior of Fully Expanded Commercially Available Endovascular Coronary Stents. Tex Heart Inst J 2011;38(5):491-501).
[2]J. Tambaca, M. Kosor, S. Canic, and D. Paniagua. Mathematical Modeling of Endovascular Stents. SIAM J Appl Math. Volume 70, Issue 6, pp. 1922-1952 (2010)

[3]S. Canic, J. Tambaca. "Cardiovascular Stents as PDE Nets: 1D vs. 3D." IMA J. Appl. Math. 77(6): pp 748-770, 2012.

First order discretization and adjoint

Consider the adjoint state

$$\int_\Omega [\hat v \cdot \frac{v^m - v^{m+1}}{\delta t} - \hat v \times \nabla \times u^{m-1} \cdot v^m - u^{m+1} \times \nabla \times \hat v \cdot v^{m+1}$$
$$+\nu \nabla \times v^m \cdot \nabla \times \hat v + \nabla \hat q \cdot v^m - q^m \nabla \cdot \hat v] + \int_\Sigma \delta t b v^m \cdot \hat v = \int_\Omega F'(p^m) \hat q$$

for all $\hat v, \hat q$ such that $\hat v \times n = 0$ on $\partial\Omega$.
Letting $\hat v = \delta u^m, \hat q = \delta p^m$ and summing in m, from 1 to M gives

$$\sum_1^M \int_\Omega F'(p^m)\delta p^m \delta t = \sum_1^M \delta t \int_\Omega [\delta u^m \cdot \frac{v^{m-1} - v^m}{\delta t} + \nu \nabla \times v^m \cdot \nabla \times \delta u^m]$$
$$+ \sum_1^M \delta t \int_\Omega (-\delta u^m \times \nabla \times u^{m-1} \cdot v^{m-1} - u^{m+1} \times \nabla \times \delta u^m \cdot v^m)$$
$$+ \sum_1^M \delta t \int_\Omega \left(\nabla \delta p^m \cdot v^m - q^m \nabla \cdot \delta u^m] + \int_\Sigma \delta t b v^m \cdot \delta u^m \right)$$

Optimality Conditions

As $\delta u^0 = 0$ and by choosing $v^M = 0$ it is also

$$\delta J = \sum_0^{M-1} \delta t \int_\Omega \left(v^m \frac{\delta u^{m+1} - \delta u^m}{\delta t} + \nu \nabla \times v^{m+1} \cdot \nabla \times \delta u^{m+1} \right)$$
$$- \sum_0^{M-1} \delta t \int_\Omega (\delta u^{m+1} \times \nabla \times u^m \cdot v^m + u^{m+1} \times \nabla \times \delta u^m \cdot v^m)$$
$$- \sum_0^{M-1} \delta t \left(\int_\Omega [\delta p^{m+1} \nabla \cdot v^{m+1} + q^{m+1} \nabla \cdot \delta u^{m+1}] + \int_\Sigma (\delta t b \delta u^{m+1} + \delta p^{m+1} n) \cdot v^{m+1} \right)$$

The same is found by linearizing (1) and taking $\hat{u} = v^m$, $\hat{q} = q^m$, except that there is an additional term due to δb. In fine

$$\delta J = -\delta t^2 \int_\Sigma \delta b \left(\sum_0^{M-1} u^{m+1} \cdot v^m \right)$$

Optimality Conditions

As $\delta u^0 = 0$ and by choosing $v^M = 0$ it is also

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$$- \sum_0^{M-1} \delta t \int_\Omega (\delta u^{m+1} \times \nabla \times u^m \cdot v^m + u^{m+1} \times \nabla \times \delta u^m \cdot v^m)$$
$$- \sum_0^{M-1} \delta t \left(\int_\Omega [\delta p^{m+1} \nabla \cdot v^{m+1} + q^{m+1} \nabla \cdot \delta u^{m+1}] + \int_\Sigma (\delta t b \delta u^{m+1} + \delta p^{m+1} n) \cdot v^{m+1} \right)$$

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$$\delta J = -\delta t^2 \int_\Sigma \delta b \left(\sum_0^{M-1} u^{m+1} \cdot v^m \right)$$

Preliminary Computer Experiments

With P^1-bubble/P^1 and Euler implicit scheme, starting with b=200, after 3 iterations of steepest descent with fixed step size 50, the following is found.

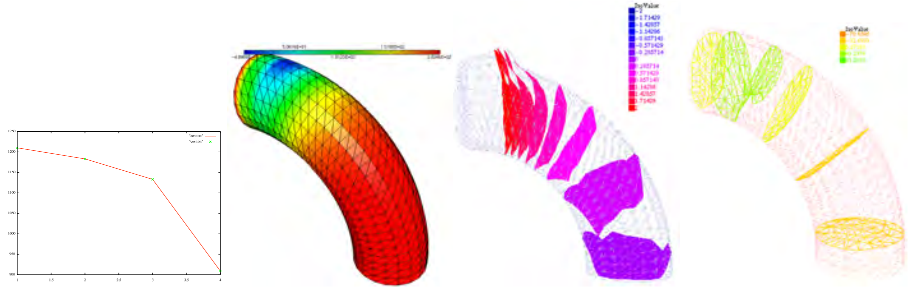


Figure : *Extreme left: Optimization criteria versus iteration number. Left: the coefficient $b(x)$ after 3 iterations. Right: effect of the change of b on the dilatation of the vessel. Extreme right: a snaptshot of the adjoint pressure.*

Preliminary test only, with $F = p^4$; only 10 time iterations with $\delta t = 0.1$. Eventually t flow is stored on disk at every time step for reuse.

Stent Mesh (by F. Hecht)

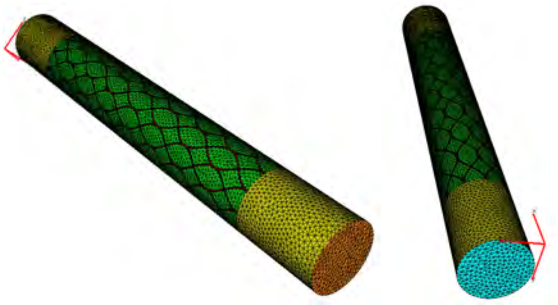


Figure : Regions: in (x=0,blue), out (x=L,orange), stent (red), cylinder off stent (green), buffer before and after (yellow). Dimensions $R = 1$, $L = C_l(N_L + N_R + N_{LL})$ where the length in axial direction of the cell is $C_l = \frac{2\pi R L_{cp}}{N_R H_{cp}}$ with $N_L = 8$ cells in axial direction, $N_R = 10$ vells in radial direction, $N_{LL} = 2$ cells before and also after the stent (buffer). $L_{cp} = 2$ (resp $H_{cp} = 2$ is the width (resp height) of the stant cell.

Preliminary Computer Experiments with Hard Stent

With P^1-bubble/P^1 and Euler implicit scheme, starting with b=200, after 3 iterations of steepest descent with fixed step size 50, the following is found.

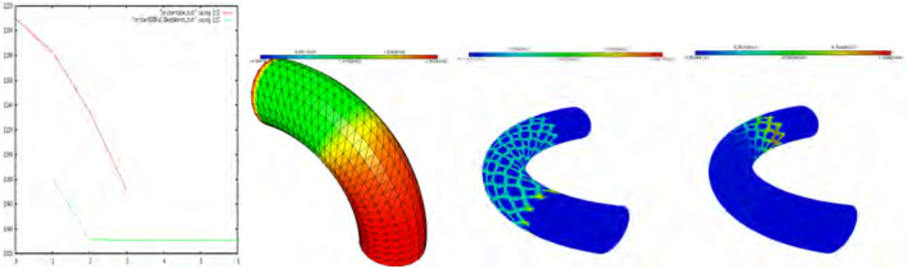
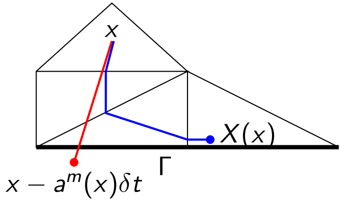


Figure : *Extreme left: Optimization criteria* $\int_\Sigma p^4$ *versus iteration number. Left: the coefficient* $b(x)$ *after 6 iterations. Right: effect of the change of b on the dilatation of the vessel. Extreme right: a snaptshot of the adjoint pressure.*

Galerkin - Characteristic Method (I)

Don't upwind or if you do, use this:



$$\partial_t u + a \cdot \nabla u|_{x,(m+1)\delta t} = \frac{u^{m+1}(x) - u^m(x - a^m(x)\delta t)}{\delta t} + O(\delta t)$$
$$= \frac{u^{m+1} - u^m oX}{\delta t} + O(\delta t)$$

with $X(x) = \mathcal{X}_{a^m}(m\delta t)$ and

$$\frac{d\mathcal{X}}{d\tau}(\tau) = a^m(\mathcal{X}(\tau)), \quad \mathcal{X}((m+1)\delta t) = x$$

Second order approximation

$$\partial_t u + a \cdot \nabla u|_{x,(m+1)\delta t} \approx \frac{3u^{m+1}(x) - 4u^m(x - a^m(x)\delta t) + u^{m-1}((x - 2a^m(x)\delta t))}{2\delta t}$$
$$= \frac{3u^{m+1} - 4u^m oX^*_{\delta t} + u^{m-1} oX^*_{2\delta t}}{2\delta t} + O(\delta t^2)$$

with $X^*_{k\delta t}(x) = \mathcal{X}_{a^{*\ m+\frac{1}{2}}}(k\ m\delta t),\ k = 1, 2$

and $a^{*\ m+\frac{1}{2}} = 2a^m - a^{m-1}$

Galerkin - Characteristic Method (II)

Zhiyong Si's modified artificial viscosity[1]

$$\boxed{\partial_t u + a\cdot\nabla u - \nu\Delta u = 0,\quad u(0),\quad u|_\Gamma \text{ given}}$$

- Step 1 $\quad \dfrac{3u^{m+\frac{1}{2}} - 4u^m o X^*_{\delta t} + u^{m-1} o X^*_{2\delta t}}{2\delta t} - (\nu + \sigma h)\Delta u^{m+\frac{1}{2}} = 0$
- Step 2 $\quad \dfrac{3u^{m+1} - 4u^m o X^*_{\delta t} + u^{m-1} o X^*_{2\delta t}}{2\delta t} - (\nu + \sigma h)\Delta u^{m+1} + \sigma h \Delta u^{m+\frac{1}{2}} = 0$

Theorem After discretization with a finite element method of order k,

$$\|u^{m+1} - u_h^{m+1}\|_0 \le C(\delta t^2 + h^{k+1} + \sigma^2 h^2 + \delta t \sigma h)$$
$$\left(\nu\delta t \sum_{j\le m}\|u^{m+1} - u_h^{m+1}\|_0^2\right)^{\frac{1}{2}} \le C(\delta t^2 + h^k + \sigma^2 h^2 + \delta t\sigma h)$$

And for N.S. $\left(\delta t\sum_{j\le m}\|p^m - p_h^m\|\right)^{\frac{1}{2}} \le C(\delta t^2 + h^k + h^2 + \delta t^2 h).$

[1] Zhiyong Si. Second order modified method of characteristics mixed defect-correction finite element method for time dependent Navier-Stokes problems. Numer Algor (2012) 59:271-300.

Galerkin-Characteristics (III)

Estimates are destroyed by quadrature error $I = \int_\Omega u_h^m(X(x))w_h(x)\mathrm{d}x$. Only estimate known is for quadrature at 3 vertices q_i^j of triangle T_j

$$I \approx \sum_j \sum_{i=1,2,3} u_h^m(X(q_i^j))w_h(q_i^j)\frac{|T_j|}{3} \Rightarrow \|u-u_h\|_\infty \le C_\epsilon(h+\delta t+\frac{h^{2-\epsilon}}{\delta t})$$

In practice a Gauss quad of degree 5 works fine. Correction to be exactly conservative c by J. Rappaz, (also tested with freefem++).

In the end one solve at each time step a generalized Stokes problem independent of time.

[1] OP and M.Tabata. Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type Int. J. Numer. Meth. Fluids 2010; 64:1240-1253

[2] J. Rappaz, S. Flotron Numerical conservation schemes for convection-diffusion equations (to appear)

Conclusion and Perspectives

1. We have studied mathematically and numerically an FSI algorithm.
2. Study the regularity of $p|_\Gamma$ for the $[u, p]$ model (done with Chacon-Girault-Murat)
3. The stability of this Surface Pressure based algorithm is its best asset.
4. It is very well suited to Optimal Shape Design of stents.
5. `freefem++` is useful for hemodynamics to prototype new ideas.

Many things to do:

1. Compare with full model on test cases.
2. A full scale numerical study
3. Validate a Chorin-Rannacher decomposition

Thanks for the Invitation